

CONTINUITY OF GENERALIZED WAVE MAPS ON THE SPHERE

DANIEL DA SILVA

ABSTRACT. We consider a generalization of wave maps based on the Adkins-Nappi model of nuclear physics. In particular, we show that solutions to this equation remain continuous at the origin, which is a first step towards establishing a regularity theory for this equation.

1. INTRODUCTION

Let $\phi : \mathbb{R}^{1+n} \rightarrow N$, where (\mathbb{R}^{1+n}, g) is the $1+n$ dimensional Minkowski spacetime with metric $\eta = \text{diag}(-1, 1, \dots, 1)$, and (N, h) is a Riemannian manifold. We say that ϕ is a *wave map* if it is a formal critical point of the action

$$S = \frac{1}{2} \int f^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j h_{ij}(u) \, dxdt = \frac{1}{2} \int f^{\mu\nu} S_{\mu\nu} \, dxdt. \quad (1)$$

Here and throughout, we use the Einstein summation convention. Critical points of (1) satisfy the wave maps equation,

$$\square \phi^i + \Gamma_{jk}^i(\phi) \partial^\alpha \phi^j \partial_\alpha \phi^k = 0. \quad (2)$$

Here the Γ_{jk}^i are the Christoffel symbols corresponding to h .

Of particular interest to physicists is the case where $n = 3$ and $N = \mathbb{S}^3$. This is known to particle physicists as the *nonlinear σ model*, first proposed by Gell-Mann and Lévy as a model for interactions of particles known as *pions*. This model, however, proved to be inadequate, due to a result known as Derrick's theorem [2], which implies the non-existence of topological solitons (static, localized solutions) for the nonlinear σ model. As a result, several alternative models were proposed, each with a different approach to avoiding Derrick's theorem.

One of these models is known as the Adkins-Nappi model [1]. In this model, we again assume $n = 3$ and $N = \mathbb{S}^3$. Derrick's theorem is avoided, however, by introducing additional terms to the action, which correspond to interactions between pions and other particles known as *vector mesons*. The new action is given by

$$S = \frac{1}{2} \int f^{\mu\nu} S_{\mu\nu} \, dxdt + \frac{1}{4} \int F^{\mu\nu} F_{\mu\nu} \, dg - \int A_\mu j^\mu \, dxdt. \quad (3)$$

Here, $A = A_\mu dx^\mu$ is a gauge potential representing the vector mesons, F is its associated electromagnetic field, given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

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while j is the baryonic current

$$j^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu \phi^i \partial_\rho \phi^j \partial_\sigma \phi^k \epsilon_{ijk}.$$

This model was recently studied by Geba and Rajeev, who began a program to prove the existence of global smooth solutions. As a first step, they focused on the equivariant case, for which solutions take the special form

$$\phi(t, x) = \phi(t, r, \omega) = (u(t, r), \omega),$$

where $\omega \in \mathbb{S}^2$. A simple computation will show that the Euler-Lagrange equations for ϕ and A decouple, and that the equations for the components of ω are trivially satisfied. The equation for u becomes

$$u_{tt} - u_{rr} - \frac{2}{r} u_r + \frac{\sin 2u}{r^2} + \frac{(u - \sin u \cos u)(1 - \cos 2u)}{r^4} = 0. \quad (4)$$

Geba and Rajeev showed in [3] that local solutions to the Cauchy problem for (4) are continuous under the assumption of smooth data of finite energy. The goal of the present work is to generalize this result for more general nonlinearities, and for all dimensions $n \geq 2$.

To motivate our result, consider (2) for $N = \mathbb{S}^n$, with $n \geq 2$. If we also make the ansatz $\phi(t, r, \omega) = (u(t, r), \omega)$, with $\omega \in \mathbb{S}^n$, it is easily seen that, for the u coordinate, (2) becomes

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r + \frac{n-1}{2} \frac{\sin 2u}{r^2} = 0.$$

We will consider a generalization based of this equation based on (4). In the case $n = 3$, the additional terms of the Adkins-Nappi Lagrangian introduced nonlinearities of the form

$$\frac{f(u)f'(u)}{r^\alpha},$$

where $f(0) = 0$. This leads us to consider the Cauchy problem

$$\begin{cases} u_{tt} - u_{rr} - \frac{n-1}{r} u_r + \frac{n-1}{2} \frac{\sin 2u}{r^2} + \frac{f(u)f'(u)}{r^\alpha} = 0 \\ u(t_0) = u_0 \\ u_t(t_0) = u_1, \end{cases} \quad (5)$$

where u_0, u_1 are C^∞ functions.

2. FORMULATION AND STATEMENT OF RESULTS

We begin our analysis with the observation that equation (5) is a semilinear hyperbolic equation. As such, the local existence theory assures us that smooth solutions exist, at least for a finite time T_0 . Our ultimate goal is to show that the conserved energy for (5), given by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} u_t^2 + u_r^2 + (n-1) \frac{\sin^2 u}{r^2} + \frac{f^2(u)}{r^\alpha} dx,$$

does not concentrate. Due to the symmetry of the problem, if the energy does concentrate, then it can only do so at $r = 0$. This indicates that the origin requires special attention.

As a first step towards proving non-concentration, we will show that solutions remain continuous at the origin. Since (5) is invariant under translations and

reflections in time, we can consider the Cauchy problem where the data is given at time $t_0 = T_0 > 0$, and the possible blow-up occurs when $t = 0$.

To understand how such a result will be proven, we must examine the conserved energy for (5) more closely. Due to the presence of the terms

$$\int_{\mathbb{R}^n} \frac{\sin^2 u}{r^2} dx \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{f^2(u)}{r^\alpha} dx$$

in the expression for the energy, we must impose the boundary condition $u(t, 0) = 0$ for all t to ensure that we have solutions of finite energy. Based on this, we see that we need only show that

$$\lim_{(t,r) \rightarrow (0,0)} u(t, r) = 0.$$

Due to the finite speed of propagation, it suffices to consider (t, r) in the forward light cone

$$\Omega = \{(t, r) : 0 \leq r \leq t, 0 < t \leq T\}.$$

Thus, in the subsequent analysis, when we say that u is a smooth local solution to (5), we mean that u is a solution to (5) which is smooth in Ω .

With these facts in mind, we can now state the main theorem that will be proved.

Theorem 1. *If u is a smooth local solution of finite energy to (5), where*

$$\alpha \geq \max\{2(n-1), n+1\}$$

and

$$f(0) = 0, \quad f'(0) \neq 0, \quad f(u)f'(u)u \geq 0, \quad \int_0^\infty |f(v)| dv = \infty,$$

then

$$\lim_{T \rightarrow 0+} \int_{\Sigma_T} \left(1 - \frac{r}{t}\right) (u_t - u_r)^2 + (u_t + u_r)^2 + \frac{\sin^2 u}{r^2} + \frac{f^2(u)}{r^\alpha} = 0. \quad (6)$$

Moreover, u is continuous at the origin:

$$\lim_{(t,r) \rightarrow (0,0)} u(t, r) = 0.$$

3. PROOF OF THEOREM 1

The proof of Theorem 1 is based on the formalism of Shatah and Tahvildar-Zadeh in [5] for the study of wave maps. This formalism makes use of the method of multipliers, first introduced by Friedrichs and later developed by Morawetz in [4]. We will begin by introducing notation and basic results, which will be used in subsequent sections to prove the main portions of the result.

3.1. Preliminaries. We start by introducing the notation

$$e^\pm(t, r) = \frac{1}{2} (u_t^2(t, r) + u_r^2(t, r)) \pm \frac{n-1}{2} \frac{\sin^2 u(t, r)}{r^2} \pm \frac{f^2(u(t, r))}{2r^\alpha},$$

$$m(t, r) = u_t(t, r)u_r(t, r).$$

The quantity e^+ is often referred to as the energy density, while m is often called the momentum density. Multiplying (5) by $r^{n-1}(au_t + bu_r + cu)$ and rewriting the result as a divergence, we obtain the main differential identity which will be used in our computations.

Lemma 2. *If u is a smooth solution to (5), then it satisfies the following identity:*

$$\begin{aligned}
& \partial_t [r^{n-1}(a e^+ + b m + c u u_t)] - \partial_r [r^{n-1}(a m + b e^- + c u u_r)] \\
&= \frac{r^{n-1}}{2} \left(a_t - b_r - (n-1) \frac{b}{r} + 2c \right) u_t^2 \\
&+ \frac{r^{n-1}}{2} \left(a_t - b_r + (n-1) \frac{b}{r} - 2c \right) u_r^2 \\
&+ r^{n-1} \left(a_t + b_r + (n-3) \frac{b}{r} \right) \frac{n-1}{2} \frac{\sin^2 u}{r^2} \\
&+ r^{n-1} \left(a_t + b_r + (n-1-\alpha) \frac{b}{r} \right) \frac{f^2(u)}{2r^\alpha} \\
&+ r^{n-1} (b_t - a_r) u_t u_r + r^{n-1} u (c_t u_t - c_r u_r) \\
&- r^{n-1} c u \left(\frac{n-1}{2} \frac{\sin 2u}{r^2} + \frac{f'(u)f(u)}{r^\alpha} \right).
\end{aligned} \tag{7}$$

This identity will be used with integrals over the following subsets of \mathbb{R}^{1+n} (see Figure 1).

$$K(S, T) = \{(t, r) : 0 \leq r \leq t, 0 < S \leq t \leq T \leq T_0\}$$

$$\Sigma_T = \{(T, r) : 0 \leq r \leq T\}$$

$$C(S, T) = \{(t, t) : 0 < S \leq t \leq T \leq T_0\}$$

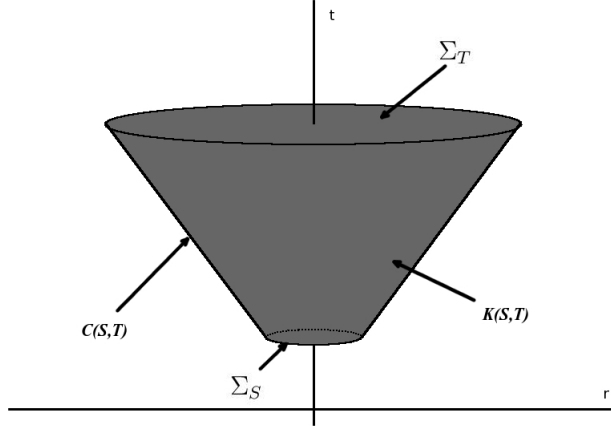


FIGURE 1. The truncated forward light cone.

Next, we would like to prove some preliminary lemmas which will be used below. Consider $(a, b, c) = (1, 0, 0)$ in (7), from which we deduce the energy differential identity

$$\partial_t (r^{n-1} e^+) - \partial_r (r^{n-1} m) = 0. \tag{8}$$

This will be used to relate the local energy

$$E(T) = \int_{\Sigma_T} e^+(t, r),$$

and the flux,

$$F(S, T) = \frac{1}{\sqrt{2}} \int_{C(S, T)} (e^+ + m)(t, r),$$

via the following lemma.

Lemma 3. *For smooth local solutions of finite energy and $S \leq T \leq T_0$, we have*

$$E(T) - E(S) = F(S, T), \quad (9)$$

which implies that the energy is monotone, while

$$\lim_{T \rightarrow 0^+} F(T) = 0, \quad (10)$$

with

$$F(T) = \lim_{S \rightarrow 0^+} F(S, T).$$

Proof. We can obtain (9) immediately by integrating the energy differential identity on the frustum $K(S, T)$, then applying the divergence theorem. For the second statement, first we note that

$$e^+ + m = \frac{(u_t + u_r)^2}{2} + \frac{n-1}{2} \frac{\sin^2 u}{r^2} + \frac{f^2(u)}{2r^\alpha} \geq 0.$$

It follows that $F(S, T) \geq 0$ for all $T \geq S \geq 0$. Combining this fact with (9), we may conclude that

$$E(T) \geq E(S).$$

For the final statement, we note that the monotonicity of E , combined with (9), tells us that $F(S, T)$ is uniformly bounded in S . Since the integrand in the definition of $F(S, T)$ is positive, we can conclude that the limit

$$\lim_{S \rightarrow 0^+} F(S, T)$$

exists. By dominated convergence, it follows that

$$\lim_{T \rightarrow 0^+} F(T) = 0.$$

□

3.2. Boundedness and Continuity Arguments. We start this section by introducing the Bogomolny functional

$$I(w) = \int_0^w |f(v)| \, dv,$$

which, based on the hypotheses on f , listed in Theorem 1, has the following properties

- $I = I(w)$ is continuous;
- $I(w) \cdot w \geq 0$, with equality only at $w = 0$;
- $|I(w)| \rightarrow \infty$ as $|w| \rightarrow \infty$.

For u a local smooth solution of (5), the fundamental theorem of calculus leads us to

$$\begin{aligned} I(u(t, r)) &= \int_0^r I'(u(t, s)) u_r(t, s) \, ds \\ &= \int_0^r |f(u(t, s))| u_r(t, s) \, ds, \end{aligned}$$

due to $u(t, 0) = 0$. Following this with an application of the Cauchy-Schwarz inequality and relying on $\alpha \geq 2(n-1)$, we deduce

$$\begin{aligned}
|I(u(t, r))| &= \left| \int_0^r |f(u(t, s))| u_r(t, s) ds \right| \\
&\leq \left(\int_0^r \frac{f^2(u(t, s))}{s^\alpha} s^{\alpha-2(n-1)} s^{n-1} ds \right)^{\frac{1}{2}} \left(\int_0^r u_r^2(t, s) s^{n-1} ds \right)^{\frac{1}{2}} \\
&\leq r^{\frac{\alpha-2(n-1)}{2}} \left(\int_0^r \frac{f^2(u(t, s))}{s^\alpha} s^{n-1} ds \right)^{\frac{1}{2}} \left(\int_0^r u_r^2(t, s) s^{n-1} ds \right)^{\frac{1}{2}} \quad (11) \\
&\lesssim r^{\frac{\alpha-2(n-1)}{2}} E^{\frac{1}{2}}(t) \left(\int_0^r \frac{f^2(u(t, s))}{s^\alpha} s^{n-1} ds \right)^{\frac{1}{2}} \lesssim r^{\frac{\alpha-2(n-1)}{2}} E(t).
\end{aligned}$$

This inequality will serve several purposes below. First, we will use it to show that u is bounded in the forward light cone.

Lemma 4. *If u is a local smooth solution to (5) of finite energy, then $u \in L^\infty(K(0, T_0))$.*

Proof. Taking advantage of the monotonicity of the energy proved in Lemma 3, we can write for all $(t, r) \in K(0, T_0)$:

$$|I(u(t, r))| \lesssim r^{\frac{\alpha-2(n-1)}{2}} E(t) \leq T_0^{\frac{\alpha-2(n-1)}{2}} E(T_0).$$

However, by construction, we know that $|I(w)| \rightarrow \infty$ as $|w| \rightarrow \infty$. This obviously implies the desired conclusion. \square

Secondly, if we write (11) for the case $n = 2$, we obtain

$$|I(u(t, r))| \lesssim r^{\frac{\alpha-2}{2}} E(t),$$

which proves the continuity of the solution, based on $\alpha \geq \max\{2(n-1), n+1\} = 3$ and the properties of the I functional. The case $n \geq 3$, however, is more subtle. For this case, we will require that part of the energy does not concentrate, as indicated in the following lemma.

Lemma 5. *If u a local smooth solution to (5) of finite energy and*

$$\lim_{T \rightarrow 0^+} \int_{\Sigma_T} \frac{f^2(u)}{r^\alpha} = 0, \quad (12)$$

then

$$\lim_{\substack{(t,r) \rightarrow (0,0) \\ (t,r) \in K(0,T_0)}} u(t, r) = 0.$$

Proof. Using the last line of (11) and the monotonicity of the energy, we infer

$$\begin{aligned}
|I(u(t, r))| &\lesssim r^{\frac{\alpha-2(n-1)}{2}} E^{\frac{1}{2}}(t) \left(\int_0^r \frac{f^2(u(t, s))}{s^\alpha} s^{n-1} ds \right)^{\frac{1}{2}} \\
&\lesssim r^{\frac{\alpha-2(n-1)}{2}} E^{\frac{1}{2}}(T_0) \left(\int_{\Sigma_t} \frac{f^2(u)}{s^\alpha} \right)^{\frac{1}{2}},
\end{aligned}$$

which, based on the hypothesis on α and properties of the Bogomolny functional, finishes the proof. \square

Thus, for $n \geq 3$, Theorem 1 will follow from Lemma 5. The remainder of this paper will be devoted to proving the limit in (12).

3.3. Main Argument. We will now complete the proof of Theorem 1 by proving the limit in (6). We start by writing (7) for $a = t$, $b = r$, and $c = (n-1)/2$:

$$\begin{aligned} & \partial_t \left[r^{n-1} \left(te^+ + rm + \frac{n-1}{2} uu_t \right) \right] - \partial_r \left[r^{n-1} \left(tm + re^- + \frac{n-1}{2} uu_r \right) \right] \\ &= r^{n-1} \left[\frac{(n-1)^2}{2} \frac{\sin^2 u}{r^2} + (n+1-\alpha) \frac{f^2(u)}{2r^\alpha} \right. \\ & \quad \left. - \frac{n-1}{2} u \left(\frac{n-1}{2} \frac{\sin(2u)}{r^2} + \frac{f'(u)f(u)}{r^\alpha} \right) \right] \end{aligned} \quad (13)$$

Integrating this over the region $K(S, T)$ yields

$$\begin{aligned} & \int_{K(S, T)} \frac{(n-1)^2}{4} \frac{uf'(u)f(u)}{r^\alpha} + (\alpha - (n+1)) \frac{f^2(u)}{2r^\alpha} \\ & + \int_{\Sigma_T} Te^+ + ru_t u_r + \frac{n-1}{2} uu_t = \int_{\Sigma_S} Se^+ + ru_t u_r + \frac{n-1}{2} uu_t \\ & + \int_{K(S, T)} \frac{(n-1)^2}{2} \frac{\sin^2 u}{r^2} - \frac{(n-1)^2}{4} \frac{u \sin(2u)}{r^2} \\ & + \frac{1}{\sqrt{2}} \int_{C(S, T)} t(e^+ + m) + r(e^- + m) + \frac{n-1}{2} u(u_t + u_r) \end{aligned} \quad (14)$$

Our goal will be to show that

$$\lim_{T \rightarrow 0^+} \frac{1}{T} \int_{\Sigma_T} Te^+ + ru_t u_r = 0. \quad (15)$$

This implies (6), which, based on Lemma 5, provides us also with the continuity of u . To achieve (15), we will prove that we can take $S = 0$ in (14), divide the resulting identity by T , and finally take the limit as $T \rightarrow 0$.

3.4. The surface integral. For $(t, r) \in C(S, T)$, we notice that

$$0 \leq t(e^+ + m) + r(e^- + m) = t(u_t + u_r)^2 \lesssim t(e^+ + m) \quad (16)$$

Based on this, we prove:

Lemma 6. *If u is a local smooth solution to (5) of finite energy, then*

$$\int_{C(S, T)} t(e^+ + m) + r(e^- + m) + \frac{n-1}{2} |u(u_t + u_r)| \lesssim TF(T) + T^{\frac{n}{2}} F(T)^{\frac{1}{2}}. \quad (17)$$

Proof. From (16) and the definitions of $F(S, T)$, respectively $F(T)$, it is easy to see that

$$0 \leq \int_{C(S, T)} t(e^+ + m) + r(e^- + m) \lesssim TF(S, T) \leq TF(T).$$

For the last integrand in (17), we can apply Cauchy-Schwarz inequality combined with the boundedness of u to obtain

$$\int_{C(S, T)} |u(u_t + u_r)| \lesssim \left(\int_{C(S, T)} u^2 \right)^{\frac{1}{2}} \left(\int_{C(S, T)} (u_t + u_r)^2 \right)^{\frac{1}{2}} \lesssim (T^n - S^n)^{\frac{1}{2}} F(S, T)^{\frac{1}{2}},$$

which provides the desired conclusion. \square

We can immediately infer, using also the decay of flux (10),

$$\lim_{T \rightarrow 0} \frac{1}{T} \left[\lim_{S \rightarrow 0} \int_{C(S,T)} t(e^+ + m) + r(e^- + m) + \frac{n-1}{2} u(u_t + u_r) \right] = 0, \quad (18)$$

which finishes this analysis.

3.5. The volume integrals. First, for the volume integral on the left-hand side of (14), we notice that the hypotheses on g and α guarantees that both integrands are positive and so

$$\begin{aligned} \lim_{S \rightarrow 0} \int_{K(S,T)} \left(\frac{n-1}{2} \right)^2 \frac{u f'(u) f(u)}{r^\alpha} + (\alpha - (n+1)) \frac{f^2(u)}{2r^\alpha} \\ = \int_{K(0,T)} \left(\frac{n-1}{2} \right)^2 \frac{u f'(u) f(u)}{r^\alpha} + (\alpha - (n+1)) \frac{f^2(u)}{2r^\alpha} \end{aligned}$$

Next, for the right-hand side integral, we can prove:

Lemma 7. *If u is a local smooth solution to (5) of finite energy and T is sufficiently small, then*

$$\int_{K(S,T)} \frac{\sin^2 u + |u \sin(2u)|}{r^2} \lesssim \begin{cases} T^{\frac{\alpha}{2}} & n = 2, \\ T^{n-1} & n \geq 3. \end{cases} \quad (19)$$

Proof. The easier of the two cases is when $n \geq 3$, for which the boundedness of u implies

$$\begin{aligned} \int_{K(S,T)} \frac{\sin^2 u + |u \sin(2u)|}{r^2} &= A(\mathbb{S}^{n-1}) \int_S^T \int_0^t \frac{\sin^2 u + |u \sin(2u)|}{r^2} r^{n-1} dr dt \\ &\lesssim \int_S^T \int_0^t r^{n-3} dr dt \lesssim T^{n-1} \end{aligned}$$

The case $n = 2$ is more subtle, as the continuity of u , obtained previously in Section 3.2, plays an important role. First, as

$$\lim_{(t,r) \rightarrow (0,0)} u(t,r) = 0,$$

$g(0) = 0$, and $f'(0) \neq 0$, it follows that, for $(t,r) \in K(S,T)$ and T sufficiently small,

$$u^2 \sim u \sin(2u) \sim \sin^2 u \quad \text{and} \quad f(u) \sim u.$$

This allows us to deduce, based on the Cauchy-Schwarz inequality,

$$u^2(t,r) \lesssim \int_0^r |f(u(t,s))| |u_r(t,s)| ds \lesssim r^{\frac{\alpha-2}{2}} E(t).$$

For $n = 2$, $\alpha \geq 3$, which implies $\frac{\alpha-4}{2} \geq -\frac{1}{2}$. We can then estimate directly

$$\int_{K(S,T)} \frac{\sin^2 u + |u \sin(2u)|}{r^2} \lesssim \int_S^T \int_0^t \frac{u^2}{r^2} r dr dt \lesssim \int_S^T \int_0^t r^{\frac{\alpha-4}{2}} dr dt \lesssim T^{\frac{\alpha}{2}}.$$

□

3.6. The time-slice integrals. We make first the observation that

$$Se^+ + ru_t u_r = (S-r) \frac{u_t^2 + u_r^2}{2} + r \frac{(u_t + u_r)^2}{2} + S \left(\frac{n-1}{2} \frac{\sin^2 u(t, r)}{r^2} + \frac{f^2(u(t, r))}{2r^\alpha} \right)$$

which tells us that the first two integrands combine to yield a non-negative quantity. Secondly, we show:

Lemma 8. *If u is a local smooth solution to (5) of finite energy and S is sufficiently small, then*

$$\int_{\Sigma_S} Se^+ + r |u_t u_r| \lesssim SE(S), \quad (20)$$

and

$$\int_{\Sigma_S} |uu_t| \lesssim \begin{cases} S^{\frac{\alpha+2}{4}} E(S), & n = 2, \\ S^{\frac{n}{2}} E(S)^{\frac{1}{2}}, & n \geq 3. \end{cases} \quad (21)$$

Proof. For the first integrand,

$$\int_{\Sigma_S} Se^+ = SE(S),$$

while the second one can be estimated using the Cauchy-Schwarz inequality,

$$\int_{\Sigma_S} r |u_t u_r| \leq S \left(\int_{\Sigma_S} u_t^2 \right)^{\frac{1}{2}} \left(\int_{\Sigma_S} u_r^2 \right)^{\frac{1}{2}} \leq SE(S).$$

Finally, for $n \geq 3$, the boundedness of u implies

$$\begin{aligned} \int_{\Sigma_S} |uu_t| &\lesssim \left(\int_{\Sigma_S} u^2 \right)^{\frac{1}{2}} \left(\int_{\Sigma_S} u_t^2 \right)^{\frac{1}{2}} \lesssim \left(\int_0^S u^2(S, r) r^{n-1} dr \right)^{\frac{1}{2}} E^{\frac{1}{2}}(S) \\ &\lesssim S^{\frac{n}{2}} E(S)^{\frac{1}{2}}. \end{aligned}$$

The argument for the $n = 2$ case is similar to the one above, but one has to use also, as for the volume integrals, the rate of decay of u . \square

3.7. Conclusion for the proof of Theorem 1. We now have all the ingredients necessary for finishing the main argument. First, using the results contained in Lemmas 6-8, we can take the limit as $S \rightarrow 0$ in (14) to obtain

$$\begin{aligned} &\int_{K(0,T)} \frac{(n-1)^2}{4} \frac{u f'(u) f(u)}{r^\alpha} + (\alpha - (n+1)) \frac{f^2(u)}{2r^\alpha} \\ &+ \int_{\Sigma_T} Te^+ + ru_t u_r = \int_{K(0,T)} \frac{(n-1)^2 \sin^2 u}{2} - \frac{(n-1)^2}{4} \frac{u \sin(2u)}{r^2} \\ &+ \frac{1}{\sqrt{2}} \int_{C(0,T)} t(e^+ + m) + r(e^- + m) + \frac{n-1}{2} u(u_t + u_r) \\ &- \int_{\Sigma_T} \frac{n-1}{2} u u_t \end{aligned} \quad (22)$$

Next, we divide by T and take the limit as $T \rightarrow 0$. Taking advantage of (18), (19), and (21), we deduce

$$\lim_{T \rightarrow 0} \frac{1}{T} \left[\int_{K(0,T)} \frac{(n-1)^2}{4} \frac{uf'(u)f(u)}{r^\alpha} + (\alpha - (n+1)) \frac{f^2(u)}{2r^\alpha} + \int_{\Sigma_T} Te^+ + ru_t u_r \right] = 0. \quad (23)$$

Using previously made remarks concerning the positivity of integrands in (23), we conclude that (15) holds, thus finishing the proof.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NY 14627